Computing the Characteristic Polynomial of Generic Toeplitz-like and Hankel-like Matrices

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ABSTRACT

New algorithms are presented for computing annihilating polynomials of Toeplitz, Hankel, and more generally Toeplitz+ Hankel-like matrices over a field. Our approach follows works on Coppersmith's block Wiedemann method with structured projections, which have been recently successfully applied for computing the bivariate resultant. A first baby-step/giant step approach—directly derived using known techniques on structured matrices—gives a randomized Monte Carlo algorithm for the minimal polynomial of an $n \times n$ Toeplitz or Hankel-like matrix of displacement rank α using $\tilde{O}\left(n^{\omega-c(\omega)}\alpha^{c(\omega)}\right)$ arithmetic operations, where ω is the exponent of matrix multiplication and $c(2.373) \approx 0.523$ for the best known value of ω. For generic Toeplitz+Hankel-like matrices a second algorithm computes the characteristic polynomial in $\tilde{O}\left(n^{2-1/\omega}\right)$ operations when the displacement rank is considered constant. Previous algorithms required $O\left(n^2\right)$ operations while the exponents presented here are respectively less than 1.86 and 1.58 with the best known estimate for ω.

KEYWORDS

Characteristic polynomial, minimal polynomial, Toeplitz matrix, Hankel matrix, Toeplitz+Hankel-like matrix.

1 INTRODUCTION

We consider the problem of computing the minimal or the characteristic polynomial of Toeplitz-like and Hankel-like matrices, which include Toeplitz and Hankel ones. The necessary definitions about those structures are given in Section 2.

Throughout the paper $T \in \mathbb{K}^{n\times n}$ is non-singular and either Toeplitz-like or Hankel-like, where $\mathbb{K}$ is a commutative field. The structure is parameterized by the displacement rank $1 \le \alpha \le n$ of T [12, 19]. In particular a Toeplitz or a Hankel matrix has displacement rank $\alpha = 2$.

The determinant of T can be computed in $\tilde{O}\left(\alpha^{\omega-1}n\right)$ operations in $\mathbb{K}$, where $\omega \le 3$ is a feasible exponent for square $n \times n$ matrix multiplication. For the best known value of ω one can take $\omega \approx 2.373$ [1, 18]. When T has generic rank profile (the leading principal submatrices are non singular) a complexity bound $\tilde{O}\left(\alpha^2 n\right)$ for the determinant is derived from [19, Cor. 5.3.3, p. 161]. In the general case, for ensuring the rank profile one uses rank-regularization techniques initially developed in [13, 15] that lead to randomized Las Vegas algorithms assuming that the cardinality of $\mathbb{K}$ is large enough; see [19, Sec. 5.6-5.7] and [3] for detailed studies in our context. Taking advantage of fast matrix multiplication is possible using the results in [3], where fundamental matrix operations, including the determinant, are performed in time $\tilde{O}\left(\alpha^{\omega-1}n\right)$ for a wide spectrum of displacement structures. In this approach the determinant is revealed by the recursive factorization of the inverse.

The characteristic polynomial $\det(xI_n - T)$ of T is a polynomial of degree n. Using an evaluation-interpolation scheme it follows that it can be computed in $\tilde{O}\left(\alpha^{\omega-1}n^2\right)$ operations in $\mathbb{K}$. We also refer to [19, Ch. 7] for a Newton-Structured iteration scheme in time $\tilde{O}\left(\alpha^2 n^2\right)$.

For a Toeplitz or Hankel matrix these complexity bounds for computing the characteristic polynomial were quadratic; our contribution establishes an improved bound $\tilde{O}\left(n^{2-1/\omega}\right)$ for generic matrices (given in compressed form), which is sub-quadratic including when using $\omega = 3$. We build on the results of [23] where only the case of a Sylvester matrix was treated, and show that the approach can be generalized to larger displacement rank families. In particular, the Hankel-(like) case requires the use of sophisticated techniques in order to handle the Toeplitz+Hankel structure [7, 9] and its generalizations [19].

The algorithms we propose fit into the broad family of Coppersmith's block Wiedemann algorithms ; we refer to [16] for the necessary material and detailed considerations on the approach. Another interpretation in terms of structured lifting and matrix fraction reconstruction is given in [23].

From $T \in \mathbb{K}^{n\times n}$, the problem is to compute the determinant (or a divisor) of the characteristic matrix $M(x) = xI_n - T$. For $1 \le m \le n$ and well chosen projection matrices V and W in $\mathbb{K}^{n\times m}$, the principle is to reconstruct an irreducible fraction description $P(x)Q^{-1}(x)$ of $V^{\mathsf{T}}M(x)^{-1}W \in \mathbb{K}(x)^{m\times m}$, where $P, Q \in \mathbb{K}[x]^{m\times m}$, from a truncated series expansion of the fraction. The denominator matrix Q carries information on the Smith normal form of $M(x)$ [16, Thm. 2.12]. Using random V and W allows to recover the minimal polynomial of T from the largest invariant factor of $M(x)$, and for a generic matrix T the characteristic polynomial is obtained [16, 23].

The matrix Q is computed from a truncation $S^{(m)} \in \mathbb{K}[x]^{m\times m}$ of the series expansion of $V^{\mathsf{T}} M(x)^{-1} W$,

$$S^{(m)}(x) = -\sum_{k\geq 0}^{2\lceil n/m\rceil} V^{\mathsf{T}}(T^{-1})^k W x^k \quad (1)$$

using for example matrix fraction reconstruction [2, 5]. We will not detail these latter aspects in this paper since they can be found elsewhere in the literature: see [16, 23] for the general techniques involved; [22, Cor. 6.4] for the power series truncation; and [17] for alternative reconstruction possibilities. The results we need on matrix polynomials are recalled in Section 3.

We focus on the computation of the power series terms $H_k = V^{\mathsf{T}}(T^{-1})^k W$ in Eq. (1). The idea for improving the complexity bounds is to use structured projections V and W in order to speed up the computation of the expansion such as in [4, 23]. A typical choice is such that the matrix product by V and W is reduced. The central difficulty is to show that the algorithm remains correct; special choices for V and W could prevent a fraction reconstruction with appropriate cost, or give a denominator matrix Q with too little information on the invariant structure of T.

For a generic input matrix and our best exponent, in Section 5 we follow the choice of [23] and work with $V = W = X$ where $X = \begin{pmatrix} I_m & 0\end{pmatrix}^{\mathsf{T}} \in \mathbb{K}^{n\times m}$. An $n\times n$ Toeplitz or a Hankel matrix is defined by $2n-1$ elements of $\mathbb{K}$, and our algorithm is correct except on a certain hypersurface of $\mathbb{K}^{2n-1}$. The same way, a Toeplitz-like or Hankel-like matrix of displacement rank α is defined by the $2n\alpha$ coefficients of its generators, and our algorithm is correct for all values of $\mathbb{K}^{2n\alpha}$ except for a hypersurface. If T is Hankel, the matrix $M(x) = xI_n - T$ is Toeplitz+Hankel and the algorithm involves a compressed form that generalizes the use of generators associated to displacement operators [9, 19]. The algorithm computes a compressed representation of $M(x)^{-1}$ modulo $x^{2\lceil n/m\rceil+1}$, and exploits its structure to truncate it into a compressed representation of $S^{(m)}(x) = X^{\mathsf{T}} M(x)^{-1} X \mod x^{2\lceil n/m\rceil+1}$ at no cost. The parameter m can be optimised to get an algorithm using $\tilde{O}\left(n^{2-1/\omega}\right)$ operations when the displacement rank is considered constant.

Before considering the fast algorithm for the generic case, in Section 4 we consider the baby steps/giant steps algorithm of [16]. Indeed, thanks to the incorporation of fast matrix multiplication in basis structured matrix operations [3], the overall approach with dense projections V and W already allows a slight exponent improvement. Taking into account that the input matrix T is structured, a direct cost analysis of the algorithm of [16] improves on the quadratic cost for Toeplitz and Hankel matrices as soon as one takes $\omega < 3$. However it is unclear to us how to compute the characteristic polynomial in this case (see the related Open Problem 3 in [14]). The algorithm we propose is randomized Monte Carlo and we compute the minimal polynomial in $\tilde{O}\left(n^{\omega-c(\omega)}\right)$ operations with $c(\omega) = \frac{\omega-1}{5-\omega}$. For Toeplitz-like and Hankel-like matrices with displacement rank α, the cost is multiplied by $\tilde{O}\left(\alpha^{c(\omega)}\right)$.

Notation. Indices of matrix and vectors start from zero. The vectors of the n-dimensional canonical basis are denoted by $e_0^n, \ldots, e_{n-1}^n$. For a matrix M, $M_{i,j}$ denotes the coefficient (i,j) of this matrix, $M_{i,*}$, its row of index i and $M_{*,j}$ its column of index j.

2 MATERIAL FOR RANK DISPLACEMENT STRUCTURES

A wide range of structured matrices are efficiently described by the action of a displacement operator [12]. There are two types of such operators: the Sylvester operators of the form

$$\nabla_{M,N} : A \mapsto MA - AN,$$

and the Stein operators of the form

$$\Delta_{M,N} : A \mapsto A - MAN;$$

where M and N are fixed matrices. A Toeplitz matrix T is defined by $2n-1$ coefficients $t_{-n+1}, \ldots, t_{n-1} \in \mathbb{K}$ such that $T = (t_{i-j})_{i,j}$. Its image through $\Delta_{Z_n, Z_n^{\mathsf{T}}}$, where $Z_n = (\delta_{i,j+1})_{0\leq i,j\leq n-1}$ has rank 2. Similarly, a Hankel matrix H is defined by $2n-1$ coefficients $h_0, \ldots, h_{2n-2}$ such that $H = (h_{i+j})_{i,j}$ and its image through $\nabla_{Z_n, Z_n^{\mathsf{T}}}$ has rank 2.

As a generalization, the class of Toeplitz-like (resp. Hankel-like) matrices is defined [8, 19] as those matrices which image through $\Delta_{Z_n, Z_n^{\mathsf{T}}}$ (resp. $\nabla_{Z_n, Z_n^{\mathsf{T}}}$) has a bounded rank α, called the displacement rank. Lastly, any sum of a Toeplitz and a Hankel matrix, (forming the class of Toeplitz+Hankel matrices) has an image of rank 4 through the displacement operator ∇_{U_n, U_n} where $U_n = Z_n + Z_n^{\mathsf{T}}$. However, contrarily to the previous instances, this operator is no longer regular, and the low rank image does not suffice to uniquely reconstruct the initial matrix: additional data (usually a first or a last column) is required for a unique reconstruction. The class of Toeplitz+Hankel-like matrices is formed by those matrices whose image through ∇_{U_n, U_n} has a bounded rank.

2.1 Product of Structured Matrices

Proposition 2.1 ([3, Theorem 1.2]). *Let $A \in \mathbb{K}^{n\times n}$ be a Toeplitz-like or Hankel-like matrix with displacement rank α given by its generators and $B \in \mathbb{K}^{n\times m}$ be a dense matrix. The multiplication of A by B can be computed in $\tilde{O}\left(n \max(\alpha, m) \min(\alpha, m)^{\omega-2}\right)$ operations in $\mathbb{K}$.*

Proposition 2.2. *Let $A, B \in \mathbb{K}^{n\times n}$ be two Toeplitz-like matrices of displacement rank α and β respectively, then their product AB is a Toeplitz-like matrix of displacement rank at most $\alpha+\beta+1$. Furthermore, given generators for A and B w.r.t. $\Delta_{Z_n, Z_n^{\mathsf{T}}}$, one can compute generators for AB w.r.t. the same operator in $\tilde{O}\left(n(\alpha+\beta)^{\omega-1}\right)$ field operations.*

Proof. Let G_A, H_A and G_B, H_B be the generators of A and B respectively. They satisfy A-$Z_n A Z_n^{\mathsf{T}} = G_A H_A^{\mathsf{T}}$ and $B - Z_n B Z_n^{\mathsf{T}} = G_B H_B^{\mathsf{T}}$. Consequently

$$\begin{aligned} AB &= (Z_n A Z_n^{\mathsf{T}} + G_A H_A^{\mathsf{T}})(Z_n B Z_n^{\mathsf{T}} + G_B H_B^{\mathsf{T}}) \\ &= Z_n AB Z_n^{\mathsf{T}} - Z_n A_{*,n} B_{n,*} Z_n^{\mathsf{T}} + (Z_n A Z_n^{\mathsf{T}} G_B) H_B^{\mathsf{T}} \\ &\quad + G_A (H_A^{\mathsf{T}} Z_n B Z_n^{\mathsf{T}} + H_A^{\mathsf{T}} G_B H_B^{\mathsf{T}}), \end{aligned}$$

and therefore $AB - Z_n AB Z_n^{\mathsf{T}} = G_{AB} H_{AB}^{\mathsf{T}}$ for

$$\begin{aligned} G_{AB} &= \left(\begin{array}{c|c|c} G_A & Z_n A Z_n^{\mathsf{T}} G_B & -Z_n A_{*,n} \end{array}\right) \\ H_{AB} &= \left(\begin{array}{c|c|c} Z_n B^{\mathsf{T}} Z_n^{\mathsf{T}} H_A + H_B G_B^{\mathsf{T}} H_A & H_B & Z_n B_{n,*}^{\mathsf{T}} \end{array}\right), \end{aligned}$$

thus showing that AB has displacement rank at most $\alpha+\beta+1$.

Computing these generators involves applying A on a dense $n \times \beta$ matrix and B on a dense $\alpha \times n$ matrix, and computing the product of an $\alpha \times n$ by an $n \times \beta$ matrix and the product of an $\alpha \times \beta$ by a $\beta \times n$ matrix. Using [3, Theorem 1.2], these cost $\tilde{O}\left(n(\alpha+\beta)^{\omega-1}\right)$ field operations. □

Proposition 2.3. *Let $A, B \in \mathbb{K}^{n\times n}$ be two Hankel-like matrices of displacement rank α and β respectively, then their product AB is a Toeplitz-like matrix of displacement rank at most $\alpha+\beta+1$. Furthermore, given generators for A and B w.r.t. $\nabla_{Z_n, Z_n^\intercal}$, generators for AB w.r.t. $\Delta_{Z_n, Z_n^\intercal}$ can be computed in $\tilde{O}\left(n(\alpha+\beta)^{\omega-1}\right)$.*

Proof. Let G_A, H_A and G_B, H_B be the generators of A and B respectively, satisfying $Z_nA - AZ_n^\intercal = G_AH_A^\intercal$ and $Z_nB - BZ_n^\intercal = G_BH_B^\intercal$. Using a similar reasoning as for Proposition 2.2 we can deduce that $AB - Z_nABZ_n^\intercal = G_{AB}H_{AB}^\intercal$ for

$$\begin{aligned} G_{AB} &= \left(\begin{array}{c|c|c} G_A & AZ_n^\intercal G_B & A_{*,n} \end{array}\right) \\ H_{AB} &= \left(\begin{array}{c|c|c} H_BG_B^\intercal H_A - B^\intercal Z_n^\intercal H_A & H_B & B_{n,*}^\intercal \end{array}\right), \end{aligned}$$

thus showing that AB has displacement rank at most $\alpha+\beta+1$. Computing these generators again costs $\tilde{O}\left(n(\alpha+\beta)^{\omega-1}\right)$ field operations. □

Proposition 2.4. *Let $A \in \mathbb{K}^{n\times n}$ be a Toeplitz-like (resp. Hankel-like) matrix of displacement rank α, then for an arbitrary (resp. even) r, A^r is a Toeplitz-like matrix of displacement rank at most $(\alpha+1)r$ and its generators can be computed in $\tilde{O}\left(n(\alpha r)^{\omega-1}\right)$ field operations.*

Proof. Using fast exponentiation one computes A^r as:

$$A^r = \prod_{k=0}^{\lfloor \log r \rfloor} \left(A^{2^k}\right)^{l_k} \text{ where } r = \sum_{k=0}^{\log r} l_k 2^k,$$

which only requires squarings and products between matrices of the form A^{2^k}. When A is Toeplitz-like the result is a straightforward consequence of Proposition 2.2; when it is Hankel-like the product A^2 is computed using Proposition 2.3, the remaining products are between Toeplitz-like matrices, and the result again follows from Proposition 2.2. □

2.2 Reconstruction of a Toeplitz+Hankel-like Matrix from its Generators

The operator ∇_{U_n, U_n} is defined in [19, Section 4.5] as partly-regular, which means that a Toeplitz+Hankel-like matrix is completely defined by its generators *and* its *irregularity set* that contains all the entries in either its first row, its last row, its first column or its last column.

A formula to recover a dense representation of the matrix from its generators and its first column is given in [19, Theorem 4.5.1].

Theorem 2.5 ([19]). *Let $M \in \mathbb{K}^{n\times n}$ be a Toeplitz+Hankel-like matrix, $G, H \in \mathbb{K}^{n\times\alpha}$ its generators and $c_0 = Me_0^n$ its first column, then*

$$M = \tau_{U_n}(c_0) - \sum_{j=0}^{\alpha-1} \tau_{U_n}(G_{*,j})\tau_{Z_n}(Z_nH_{*,j})^\intercal \tag{2}$$

where for an $n\times n$ matrix A and a vector v of length n $\tau_A(v)$ denotes the matrix of the algebra generated by A which has v as its first column.

We show that one can derive a fast reconstruction algorithm for a Toeplitz+Hankel-like matrix from Eq. (2) and first detail the structure of the various $\tau_A(v)$ matrices.

Lemma 2.6. *$\tau_{Z_n}(v)^\intercal$ is the Toeplitz upper-triangular matrix with $v^\intercal$ as its first row.*

Lemma 2.7. *$\tau_{U_n}(v) = \sum_{i=0}^{n-1} v_iQ_i(U_n)$ where $Q_0(x) = 1$, $Q_1(x) = x$ and $Q_{i+1}(x) = xQ_i(x) - Q_{i-1}(x)$.*

Proof. The first column of $Q_i(U_n)$ is e_i^n. □

Corollary 2.8. *Column j of $\tau_{U_n}(v)$ is $Q_j(U_n)v$.*

Proof. With Lemma 2.7 and after checking the property for $j \in \{0, 1\}$, it suffices to prove $Q_i(U_n)_{*,j+1} = U_nQ_i(U_n)_{*,j} - Q_i(U_n)_{*,j-1}$. This is true for $i \in \{0, 1\}$ and if it is for i and $i-1$, then

$$\begin{aligned} Q_{i+1}(U_n)_{*,j+1} = U_n^2Q_i(U_n)_{*,j} - U_nQ_i(U_n)_{*,j-1} \\ - U_nQ_{i-1}(U_n)_{*,j} + Q_{i-1}(U_n)_{*,i-1} \end{aligned}$$

□

From these we can write the following proposition, inspired by [7, Proposition 4.2], and which enables fast recursive reconstruction of the columns of a Toeplitz+Hankel-like matrix.

Proposition 2.9. *Let $M \in \mathbb{K}^{n\times n}$ be a Toeplitz+Hankel-like matrix, $G, H \in \mathbb{K}^{n\times\alpha}$ its generators for ∇_{U_n,U_n} and $c_0 = Me_0^n$ its first column. With the notation $c_{-1} = 0$, the columns $(c_k)_{0\le k\le n-1}$ of M follow the recursion:*

$$c_{k+1} = U_nc_k - c_{k-1} - \sum_{j=0}^{\alpha-1} H_{k,j}G_{*,j}. \tag{3}$$

Proof. Let C be the matrix defined by the recursion formula and initial conditions of Proposition 2.9, we will prove $C = M$.

By definition c_0 is the first column of M; assume now that for $j \le k$, c_j is column j of M, then Eq. (3) can be detailed as

$$\begin{aligned} c_{k+1} &= Q_{k+1}(U_n)c_0 - \sum_{j=0}^{\alpha-1}\sum_{i=1}^{k-1} Q_{k+1}(U_n)G_{*,j}H_{i,j}) \\ &\quad - U_n\sum_{j=1}^{\alpha} Q_k(U_n)G_{*,j}H_{k,j} - \sum_{j=1}^{\alpha} H_{k,j}G_{*,j} \\ &= M_{*,k+1} \text{ by Eq. (2)} \end{aligned}$$

□

3 MATERIAL FOR MATRIX POLYNOMIALS

We rely on the material from [16, 23]. For matrix polymonials and fractions the reader may refer to [11]. The rational matrix $H(x) = V^\intercal M(x)^{-1}W$ over $\mathbb{K}(x)$ can be written as a fraction of two polynomial matrices. A right fraction description is given by square polynomial matrices $P(x)$ and $Q(x)$ such that $H(x) = P(x)Q(x)^{-1} \in \mathbb{K}(x)^{m\times m}$, and a left description by $P_l(x)$ and $Q_l(x)$ such that $H(x) = Q_l(x)^{-1}R_l(x) \in \mathbb{K}(x)^{m\times m}$. Degrees of denominator matrices are minimized using column-reduced forms. A non-singular polynomial matrix is said to be column-reduced if its leading column coefficient matrix is non-singular [11, Sec. 6.3]. We also have the notion of irreducible and minimal fraction descriptions. If P and Q (resp. P_l and Q_l) have unimodular right (resp. left) matrix gcd's [11, Sec. 6.3] then the description is called irreducible. If Q (resp. Q_l) is column-reduced then the description is called minimal.

For a given m, define $1 \leq \nu \leq n$ to be the sum of the degrees of the first m largest invariant factors of $M(x)$ (equivalently, the first m diagonal elements of its Smith normal form). The following will ensure that the minimal polynomial of T, which is the largest invariant factor of $M(x)$ can be computed from the Smith normal form of an appropriate denominator $Q(x)$; see Corollary 4.2.

Theorem 3.1. *([16, Thm. 2.12] and [22]) Let V and W be block vectors over a sufficiently large field $\mathbb{K}$ whose entries are sampled uniformly and independently from a finite subset $S \subseteq \mathbb{K}$. Then with probability at least $1 - 2n/|S|$, $H(x) = V^\intercal M(x)^{-1} W$ has left and right irreducible descriptions with denominators of degree $\lceil \nu/m \rceil$, of determinantal degree ν, and whose i^{th} invariant factor (starting from the largest degree) is the i^{th} invariant factor of $M(x)$.*

The next result we need is concerned with the computation of an appropriate denominator Q as soon as the truncated power series in Eq. (1) is known. We notice that $H(x) = V^\intercal M(x)^{-1} W$ is strictly proper in that it tends to zero when x tends to infinity. For fraction reconstruction we use the computation of minimal approximant bases (or σ-bases) [2, 21], and the algorithm with complexity bound $\tilde{O}\left(m^{\omega-1} n\right)$ in [5, 10].

Theorem 3.2 ([5, Lemma 3.7]). *Let $H \in \mathbb{K}(x)^{m \times m}$ be a strictly proper power series, with left and right matrix fractions descriptions of degree at most d. A denominator Q of a right irreducible description $H(x) = P(x)Q(x)^{-1}$ can be computed in $\tilde{O}\left(m^{\omega-1} n\right)$ arithmetic operations from the first $2d + 1$ terms of the expansion of H.*

In our case, from Theorem 3.1 we will obtain the existence of appropriate fractions of degree less than $\lceil n/m \rceil$, and use Theorem 3.2 for bounding the cost of the computation of Q.

4 A BABY-STEP GIANT STEP ALGORITHM

In this section, we propose a direct adaptation of the baby steps/giant steps variant of Coppersmith's block-Wiedemann algorithm from [16, Sec. 4] to the case of structured matrices. In order to compute the terms of the series (1), we will assume that the input matrix T has been inverted, using [3, Theorem 6.6]. In this section we will therefore denote by T this inverse and compute the projections of its powers.

4.1 Description of the Algorithm

Let $V, W \in \mathbb{K}^{n \times m}$ be the block vectors used for the projection. Algorithm 1 performs r baby steps and s giant steps to compute the first terms of the sequence $H_k = V^\intercal T^k W = V^\intercal (T^r)^j T^i W$ for $0 \leq k \leq 2\lceil n/m \rceil$, $0 \leq i < r$, $0 \leq j < s$ and $rs \geq k + 1$.

This algorithm relies on three main operations:

(1) the product of a structured matrix to dense rectangular matrix, supported by Proposition 2.1 for Lines 3 and 7;
(2) the exponentiation of a structured matrix, supported by Proposition 2.4 for Line 4;
(3) the product of two dense rectangular matrices for Line 8.

Algorithm 1 Compute $H_k = V^\intercal T^k W$ for $0 \leq k \leq 2\lceil n/m \rceil$

Input: Generators of $T \in \mathbb{K}^{n \times n}$, Toeplitz-like or Hankel-like
Input: $m, r, s \in \mathbb{N}$ s.t. $rs \geq 2\lceil n/m \rceil + 1$, r even if T is Hankel-like
Input: $V, W \in \mathbb{K}^{n \times m}$
Output: $H = \left(H_{rj+i}\right)_{j<s, i<r}$ where $H_k = V^\intercal T^k W$

1: $W_0 \leftarrow W$
2: **for** $1 \leq i \leq r-1$ **do**
3: $\quad W_i \leftarrow T W_{i-1}$
4: $R \leftarrow T^r$
5: $V_0 \leftarrow V$
6: **for** $1 \leq j \leq s-1$ **do**
7: $\quad V_j^\intercal \leftarrow V_{j-1}^\intercal R$
8: $H \leftarrow \begin{pmatrix} V_0 & \ldots & V_{s-1} \end{pmatrix}^\intercal \begin{pmatrix} W_0 & \ldots & W_{r-1} \end{pmatrix}$

4.2 Cost Analysis

Theorem 4.1. *Algorithm 1 runs in $\tilde{O}\left(n^{\omega - \frac{\omega-1}{5-\omega}} \alpha^{\frac{\omega-1}{5-\omega}}\right)$ operations in $\mathbb{K}$ for well chosen m, r and s.*

For instance, when the displacement rank α is constant, and with the best known estimate $\omega = 2.373$ [1] the cost becomes $\tilde{O}\left(n^{1.851}\right)$ while it is $\tilde{O}\left(n^2\right)$ for $\omega = 3$.

Proof. From Proposition 2.1, applying an $n \times m$ block to T can be done in $\tilde{O}\left(n \max(m, \alpha) \min(m, \alpha)^{\omega-2}\right)$ field operations. Hence the r baby-steps, Line 3, computing the $(T^i W)_{0 \leq i < r}$ cost overall

$$\tilde{O}\left(nr \max(m, \alpha) \min(m, \alpha)^{\omega-2}\right) \tag{4}$$

field operations.

By Proposition 2.4, the initialization of the giant steps, Line 4 computing a structured representation for T^r, can be done in

$$\tilde{O}\left(n r^{\omega-1} \alpha^{\omega-1}\right) \tag{5}$$

operations in $\mathbb{K}$.

Then each of the giant steps, Line 7, is a product of an $m \times n$ dense matrix with an $n \times n$ matrix of displacement rank αr. From Proposition 2.1, these s steps cost

$$\tilde{O}\left(ns \max(m, \alpha r) \min(m, \alpha r)^{\omega-2}\right) \tag{6}$$

Lastly, the computation of the product resulting in H, Line 8, uses $\tilde{O}\left(n \max(mr, ms) \min(mr, ms)^{\omega-2}\right)$ or equivalently

$$\tilde{O}\left(n m^{\omega-1} \max(r, s) \min(r, s)^{\omega-2}\right) \tag{7}$$

field operations.

Let $m = \left\lceil n^{\frac{\omega-3}{\omega-5}} \alpha^{\frac{2}{5-\omega}} \right\rceil$ and set $r = s = \left\lceil \sqrt{2n/m} \right\rceil$. Note that $\alpha \leq m \leq \alpha r$. Therefore (4) is dominated by (7). Moreover (6) writes $\tilde{O}\left(n^2 m^{\omega-3} \alpha\right)$, (7) writes $\tilde{O}\left(n^{\frac{\omega+1}{2}} m^{\frac{\omega-1}{2}}\right)$ and both terms equal

$$\tilde{O}\left(n^{\omega - \frac{\omega-1}{5-\omega}} \alpha^{\frac{\omega-1}{5-\omega}}\right).$$

Finally, (5) writes $\tilde{O}\left(n^{\frac{\omega+1}{2}} \left(\frac{\alpha^2}{m}\right)^{\frac{\omega-1}{2}}\right)$ and is thus dominated by (7). ◻

Let us now suppose that the entries of V and W are sampled uniformly and independently from a finite subset $S \subseteq \mathbb{K}$, we then have the following:

Corollary 4.2. *The minimal polynomial of an $n \times n$ Toeplitz-like or Hankel-like matrix with displacement rank α can be computed by a Monte Carlo algorithm in*

$$\tilde{O}\left(n^{\omega - \frac{\omega-1}{5-\omega}} \alpha^{\frac{\omega-1}{5-\omega}}\right)$$

field operations with a probability of success of at least $1 - (n^2 + 3n)/|S|$.

Proof. The first step is to compute the inverse of T, using [3, Theorem 6.6] in $\tilde{O}\left(n\alpha^{\omega-1}\right)$ operations in $\mathbb{K}$. Then running Algorithm 1 on T^{-1} costs $\tilde{O}\left(n^{\omega - \frac{\omega-1}{5-\omega}} \alpha^{\frac{\omega-1}{5-\omega}}\right)$ which dominates $\tilde{O}\left(n\alpha^{\omega-1}\right)$ since $\alpha \leq n$. From the sequence of matrices $(H_k)_{0 \leq k \leq 2n/m}$, one can compute a minimal denominator Q for $H(x) = V^{\mathsf{T}}(xI_n - T)^{-1}W \in \mathbb{K}[x]^{m \times m}$ in $\tilde{O}\left(nm^{\omega-1}\right)$ field operations, by Theorem 3.2.

Using Theorem 3.1, the minimal polynomial is then obtained as the first invariant factor in the Smith form of Q, computed by [20, Proposition 41]. This step also costs $\tilde{O}\left(nm^{\omega-1}\right)$ field operations and since $m \leq n$ we have

$$nm^{\omega-1} \leq n^{\frac{\omega+1}{2}} m^{\frac{\omega-1}{2}}$$

which shows that the cost of these last two computations will always be dominated by the cost of the product (7). The probability of failure for the computation of T^{-1} is $n(n+1)/|S|$ by [3, Lemma 6.2]. A union bound combining this probability and the failure problablity of Theorem 3.1 yields a probability of failure of $(n^2 + 3n)/|S|$. □

Note that this result carries over to the computation of the characteristic polynomial of any Toeplitz-like or Hankel-like matrix T having fewer than m invariant factors in its Frobenius normal form.

5 AN ALGORITHM BASED ON STRUCTURED INVERSION

In this section we propose an algorithm computing the determinant of a generic structured polynomial matrix $M \in \mathbb{K}[x]^{n \times n}$ with displacement rank α based on the structure of the ΣLU representation of Toeplitz-like matrix, or a generalization thereof for Hankel-like matrices, as presented in (2).

Principle of the algorithm. Here, the sequence $(H_k = V^{\mathsf{T}} T^{-k} W)_k$ is obtained as the matrix coefficients of the series expansion $V^{\mathsf{T}} M^{-1} W$. As $2\lceil n/m \rceil + 1$ terms are required, and with the special choice $V = W = X = \begin{pmatrix} I_m & | & 0 \end{pmatrix}^{\mathsf{T}}$, this boils down to computing a dense representation of the $m \times m$ leading principal submatrix of M^{-1} mod $x^{2\lceil n/m \rceil + 1}$. The outline of the algorithm is as follows.

(1) Compute the inverse $M^{-1} \mod x^{2\lceil n/m \rceil + 1}$ in a compressed representation
(2) Crop this representation to form a representation of the $m \times m$ leading principal submatrix;
(3) Extract the dense representation from this representation.

We will now present the algorithm specialized for the two classes of interest.

5.1 The Algorithm for Toeplitz-like Matrices

A Toeplitz-like matrix T is represented by a pair of generators $G, H \in \mathbb{K}^{n \times \alpha}$ satisfying $T = \sum_{i=0}^{\alpha-1} L(G_{*,i}) L(H_{*,i})^T$, where $L(v)$ is the lower triangular Toeplitz matrix with v as its first column [12, 13]. The $m \times m$ leading principal submatrix of any product $L(v)L(w)^{\mathsf{T}}$ is the product of the $m \times m$ leading principal submatrix of these factors, which in turn is $L(v_{1..m})L(w_{1..m})^{\mathsf{T}}$. Algorithm 2 relies on this property to produce $S^{(m)}$ from the m first rows of the generators of T^{-1}.

Algorithm 2 Compute $S^{(m)}$: Toeplitz-like case

Input: (G, H) generators of $M \in \mathbb{K}[x]^{n \times n}$, a Toeplitz-like matrix of displacement rank α

Output: Dense representation of $S^{(m)} = X^{\mathsf{T}} M^{-1} X \mod x^{2\lceil n/m \rceil + 1}$

1: $(E, F) \leftarrow$ generators for $M^{-1} \mod x^{2\lceil n/m \rceil + 1}$
2: $E' \leftarrow X^{\mathsf{T}} E$; $F' \leftarrow FX$
3: $S^{(m)} \leftarrow \sum_{i=0}^{\alpha-1} L(E'_{*,i}) L(F'_{*,i})^{\mathsf{T}} \mod x^{2\lceil n/m \rceil + 1}$

Theorem 5.1. *Algorithm 2 is correct for $M = xI_n - T$ and T generic and uses*

$$\tilde{O}\left(\frac{n^2}{m}\alpha^{\omega-1} + nm\alpha\right)$$

operations in $\mathbb{K}$.

Proof. From the above remark, $E' = E_{1..m,*}$ and $F' = F_{1..m,*}$ are generators for $S^{(m)} = X^{\mathsf{T}} M^{-1} X$. Note that no division by x in the ring $\mathbb{K}[x]/\langle x^{2\lceil n/m \rceil + 1} \rangle$ will occur in Line 1 as T has generic rank profile, and consequently all leading principal minors of $M(x)$ are not divisible by x which shows the correctness.

By [3, Theorem 34], Line 1, computing the generators of M^{-1}, can be computed in $\tilde{O}\left(n\alpha^{\omega-1}\right)$ operations over $\mathbb{K}[x]/\langle x^{2\lceil n/m \rceil + 1} \rangle$ which in turn is

$$\tilde{O}\left(\frac{n^2}{m}\alpha^{\omega-1}\right) \tag{8}$$

operations in $\mathbb{K}$.

The dense reconstruction of $S^{(m)}$ in Line 3 is achieved by α products of an $m \times m$ Toeplitz matrix $L(E'_{*,i})$ by an $m \times m$ dense matrix $L(F'_{*,i})^{\mathsf{T}}$ for a total cost of

$$\tilde{O}\left(nm\alpha\right) \tag{9}$$

operations in $\mathbb{K}$. □

Corollary 5.2. *The characteristic polynomial of a generic $n \times n$ Toeplitz-like matrix with displacement rank α can be computed in $\tilde{O}\left(n^{2-\frac{1}{\omega}} \alpha^{\frac{(\omega-1)^2}{\omega}}\right)$ operations in $\mathbb{K}$ when $\alpha = O\left(n^{\frac{\omega-2}{-\omega^2+4\omega-2}}\right)$, and $\tilde{O}\left(n^{\frac{3}{2}} \alpha^{\frac{\omega}{2}}\right)$ otherwise.*

Note that this is $O(n^{1.579})$ (resp. $O(n^{1.667})$) for α constant and $\omega = 2.373$ (resp. $\omega = 3$). When $\alpha = \Theta\left(n^{\frac{\omega-2}{-\omega^2+4\omega-2}}\right)$ and taking $\omega = 2.373$ (resp. $\omega = 3$), both expressions become $\tilde{O}\left(n^{1.74}\right)$ (resp. $\tilde{O}\left(n^3\right)$).

The complexity when α is low can also be written as

$$\tilde{O}\left(n^{\omega-f(\omega)}\alpha^{f(\omega)}\right),$$

similarly as in Theorem 4.1, which can be interpreted as a transfer of part of the exponent from n to α by using the structure of the matrix.

Proof. The family of Toeplitz matrices presented in Section 6.1 proves that for a generic Toeplitz-like matrix T, the matrix $\mathcal{H}^{(n)} = \mathcal{H}_{1..n,1..n}$ is non-singular, where

$$\mathcal{H} = \left(V^\mathsf{T} T^{i+j} W\right)_{0\le i,j\le \lceil n/m\rceil -1}.$$

Then [23, Lemma 2.4] implies that the irreducible left and right fractions descriptions of $X^\mathsf{T} M^{-1} X$ have degree at most $\lceil n/m\rceil$. Thus Theorem 3.2 ensures that an appropriate denominator Q of a right fraction description of $X^\mathsf{T} M^{-1} X$ can be computed from $S^{(m)} = X^\mathsf{T} M^{-1} X \mod x^{2\lceil n/m\rceil+1}$.

Besides the computation of $S^{(m)}$ by Theorem 5.1, the computation of the denominator Q of its irreducible right fraction description costs

$$\tilde{O}\left(nm^{\omega-1}\right) \tag{10}$$

operations by Theorem 3.2. Computing the determinant of Q has same cost by [6, 20]. The total cost depends on α.

Case 1: $\alpha = O\left(n^{\frac{\omega-2}{-\omega^2+4\omega-2}}\right)$. We set $m = n^{\frac{1}{\omega}}\alpha^{\frac{\omega-1}{\omega}}$ so that $\alpha = O(m^{\omega-2})$ and the term (9) is dominated by (10). For the chosen value of m the terms (8) (decreasing in m) and (10) (increasing in m) are equal, leading to a full cost of $\tilde{O}\left(n^{2-\frac{1}{\omega}}\alpha^{\frac{(\omega-1)^2}{\omega}}\right)$ operations in $\mathbb{K}$.

Case 2: $\alpha = \Omega\left(n^{\frac{\omega-2}{-\omega^2+4\omega-2}}\right)$. We set $m = n^{\frac{1}{2}}\alpha^{\frac{\omega-2}{2}}$ so that $\alpha = \Omega(m^{\omega-2})$. In this case the term (10) is dominated by (9) and for this value of m we have equality between the terms (8) and (9), leading to a full cost of $\tilde{O}\left(n^{\frac{3}{2}}\alpha^{\frac{\omega}{2}}\right)$ operations in $\mathbb{K}$. □

5.2 The Algorithm for Hankel-like Matrices

In this section we are interested in adapting the previous algorithm to Hankel-like matrices. If T is Hankel-like then $M(x) = xI_n - T$ is Toeplitz+Hankel-like.

We will thus generalize and consider that T is a Toeplitz+ Hankel-like matrix. We are interested in computing the first $2\lceil n/m\rceil + 1$ terms of the series $X^\mathsf{T} M(x)^{-1} X$. We are going to adapt the Toeplitz algorithm and use Pan's Divide-and-Conquer algorithm for inversion [19, Chapter 5]. Computing the characteristic polynomial from there does not depend on the structure of M or T.

The strategy consists in computing generators for the truncated matrix from which we can recover a dense representation. Algorithm 3 details the steps. The generators and irregularity set of the inverse in Line 1 are computed with Pan's Divide and Conquer algorithm [19], as well as the solution to the linear system. The following lines are dedicated to the reconstruction of the dense representation of $S^{(m)}(x)$ from the generators. The correctness of Algorithm 3 is proved by Proposition 2.9.

Algorithm 3 Compute $S^{(m)}$: Toeplitz+Hankel-like case

Input: (G, H, v) generators and irregularity set of $M \in \mathbb{K}[x]^{n\times n}$, a Toeplitz+Hankel-like matrix of displacement rank α.

Output: Dense representation of $S^{(m)}(x) = X^\mathsf{T} M^{-1}(x) X \mod x^{2\lceil n/m\rceil+1}$

1: $(E, F, u), c \leftarrow$ generators and irregularity set of the inverse of M, solution of $Mc = e_0^n$

2: $c_0 \leftarrow X^\mathsf{T} c$

3: $c_1 \leftarrow U_m c_0 - \sum_{i=0}^{\alpha-1} E_{0,i} F_{0\ldots m-1,i}$

4: **for** $1 \le k \le m-2$ **do**

5: $\quad c_{k+1} \leftarrow U_m c_k - c_{k-1} - \sum_{i=0}^{\alpha-1} E_{k,i} F_{0\ldots m-1,i}$

6: $S^{(m)}(x) = (c_0 || \cdots || c_{m-1})$

Theorem 5.3. *Algorithm 3 is correct for $M = xI_n - T$ and T generic and uses*

$$\tilde{O}\left(\frac{n^2}{m}\alpha^2 + mn\alpha\right)$$

operations in $\mathbb{K}$.

Proof. Line 1 can be done in $\tilde{O}(\alpha^2 n)$ operations in the base ring, so $\tilde{O}\left(\frac{n^2}{m}\alpha^2\right)$ operations on $\mathbb{K}$ [19, Corollary 5.3.3]. Each step of the `for` loop consists of a number of polynomial operations modulo $x^{2\lceil n/m\rceil+1}$ linear in $m\alpha$ as U_m has only two non-zero entries on each row. Lines 2 to 5 can be done in $\tilde{O}(m^2\alpha)$ operations in the base ring, so $\tilde{O}(nm\alpha)$ operations on $\mathbb{K}$. □

The minimal polynomial is then obtained the same way as in Section 4 which leads to Corollary 5.4.

Corollary 5.4. *The characteristic polynomial of a generic $n \times n$ Toeplitz+Hankel-like matrix with displacement rank α can be computed in $\tilde{O}\left(n^{2-\frac{1}{\omega}}\alpha^{\frac{2(\omega-1)}{\omega}}\right)$ field operations when $\alpha = O\left(n^{\frac{\omega-2}{4-\omega}}\right)$, and $\tilde{O}\left(n^{\frac{3}{2}}\alpha^{\frac{3}{2}}\right)$ otherwise.*

The complexity in n is the same as in the Toeplitz-like case but there is a stronger dependence in α as there is no known algorithm to compute the inverse of a Toeplitz+Hankel-like matrix in $O(n\alpha^{\omega-1})$, the best one depending on α^2.

Proof. The family of Hankel matrices presented in Section 6.2 now proves that for all generic Hankel-like matrix T, the matrix $\mathcal{H}^{(n)}$ is non-singular. The rest of the proof is similar to the Toeplitz-like case in Corollary 5.2.

Again the overall cost is that for computing the denominator and its determinant in $\tilde{O}\left(nm^{\omega-1}\right)$ operations in $\mathbb{K}$ plus the cost of computing the sequence H_k. We distinguish two cases:

If $\alpha = O\left(n^{\frac{\omega-2}{4-\omega}}\right)$: Setting $m = n^{\frac{1}{\omega}}\alpha^{\frac{2}{\omega}}$ so that $\alpha = O(m^{\omega-2})$ and the full cost is $\tilde{O}\left(n^{2-\frac{1}{\omega}}\alpha^{\frac{2(\omega-1)}{\omega}}\right)$.

If $\alpha = \Omega\left(n^{\frac{\omega-2}{4-\omega}}\right)$: Setting $m = n^{\frac{1}{2}}\alpha^{\frac{1}{2}}$ so that $\alpha = \Omega(m^{\omega-2})$ and the full cost is $\tilde{O}\left(n^{\frac{3}{2}}\alpha^{\frac{3}{2}}\right)$. □

6 SPECIAL MATRICES FOR GENERICITY

The generic matrices T for which our algorithms output the characteristic polynomial are matrices such that $\mathcal{H}^{(n)} = \mathcal{H}_{1..n,1..n}$ is non-singular (Corollaries 5.2 and 5.4), where

$$\mathcal{H} = \left(V^{\mathsf{T}} T^{i+j} W\right)_{0 \le i,j \le \lceil n/m \rceil - 1}$$

The first algorithm is Monte Carlo with matrices V and W sampled at random. In the second algorithm, however $V = W = X$ are fixed, and $\det \mathcal{H}^{(n)}$ is a polynomial in the coefficients of T. Toeplitz and Hankel matrices have $2n - 1$ independant coefficients. The coefficients of a Toeplitz-like or Hankel-like matrix of displacement rank α are themselves polynomials in the coefficients of its generators, so $\det \mathcal{H}^{(n)}$ is by composition a polynomial on the $2n\alpha$ coefficients of the $n \times \alpha$ generators of T.

In this section, we show that $\det \mathcal{H}^{(n)}$ is not uniformly zero on the space of Toeplitz (resp. Hankel) matrices by finding one Toeplitz (resp. Hankel) matrix for which $\mathcal{H}^{(n)}$ is non-singular. This shows the algorithm is correct for all matrices of each class except for those with coefficients in a certain variety of $\mathbb{K}^{2n-1}$. As the displacement rank of the matrices we show is 2 or less, they are Toeplitz-like (resp. Hankel-like) and can be represented with larger generators (padded with zeros). The algortihm is thus also correct for matrices with displacement rank $\alpha \ge 2$ whose generators' coefficients are not in a certain variety of $\mathbb{K}^{2n\alpha}$. Both matrices are also Toeplitz+Hankel and Toeplitz+Hankel-like so the same reasoning shows the algorithm is correct for all Toeplitz+Hankel matrices except for those with coefficients in a certain hypersurface of $\mathbb{K}^{4n-2}$ and all Toeplitz+Hankel-like matrices with displacement rank $\alpha \ge 4$ except for those on a certain hypersurface of $\mathbb{K}^{2n\alpha}$.

6.1 A Toeplitz Point

Let

$$T = \begin{pmatrix} 0 & I_m \\ -I_{n-m} & 0 \end{pmatrix}$$

and $M(x) = xI_n - T$. Let $P(x) \in \mathbb{K}[x]^{n \times m}$ defined by:

$$\begin{aligned} P_{n-m+k,k} &= 1 && \text{for } 0 \le k \le m \\ P_{i,k} &= xP_{i+m,k} && \text{for } 0 \le k \le m, 0 \le i \le n-m-1 \end{aligned}$$

With

$$D(x) = \begin{pmatrix} 0 & x^{\lfloor n/m \rfloor} I_{n \bmod m} \\ x^{\lfloor n/m \rfloor - 1} I_{-n \bmod m} & 0 \end{pmatrix}$$

we can write $P(x) = \begin{pmatrix} D(x)^{\mathsf{T}} & R(x) & I_m \end{pmatrix}^{\mathsf{T}}$. From there we have $M(x)P(x) = \begin{pmatrix} xD(x)^{\mathsf{T}} - I_m & 0 \end{pmatrix}^{\mathsf{T}}$ and thus

$$X^{\mathsf{T}} M^{-1}(x) X = X^{\mathsf{T}} P(x) \left(xD(x) - I_m\right)^{-1}.$$

That is $X^{\mathsf{T}} M^{-1}(x) X = D(x) Q^{-1}(x)$ with $Q(x) = xD(x) - I_m$. As $xI_m D(x) - I_m Q(x) = I_m$, the fraction DQ^{-1} is irreducible and

$$\det Q = \pm x^{\lfloor n/m \rfloor (n \mod m) + (\lfloor n/m \rfloor - 1)(-n \mod m)} - 1$$

from which we get $\deg \det Q = n$. By [23, Lemma 2.4], the matrix $\mathcal{H}^{(n)}$ is therefore non-singular.

6.2 A Hankel Point

Let $T_n = (I_n + Z_n^m) J_n$. For j such that $2j \le \lceil n/m \rceil - 1$, rows jm to $(j+1)m - 1$ of $T_n^{2j} X$ are I_m and the following rows are 0. This can be seen by recursively applying the band matrix $T_n^2 = Z_n^m + I_n + Z_n^m Z_n^{m\mathsf{T}} + Z_n^{m\mathsf{T}}$ to X. By applying T_n to $T_n^{2j} X$ we get that the rows $n - (j+1)m$ to $n - jm - 1$ of $T_n^{2j+1} X$ are J_m, and the preceding rows are 0.

Let K_r be the first n columns of $\left(T^0 X | \ldots | T^{\lceil n/m \rceil - 1} X\right)$. K_r is non-singular, as its columns can be permuted to get a matrix of the form

$$\begin{pmatrix} L_1^{\mathsf{T}} & 0 \\ 0 & L_2 \end{pmatrix}$$

where L_1 and L_2 are lower triangular with ones on the diagonal. As T is symmetric, K_l defined as the first n rows of

$$\left(T^{0\mathsf{T}} X | \ldots | T^{(\lceil n/m \rceil - 1)\mathsf{T}} X\right)^{\mathsf{T}}$$

is also non-singular, as well as $\mathcal{H}^{(n)} = K_l K_r$.

REFERENCES

[1] J. Alman and V. Vassilevska Williams. 2020. A Refined Laser Method and Faster Matrix Multiplication. arXiv:2010.05846 [cs.DS]

[2] B. Beckermann and G. Labahn. 1994. A uniform approach for the fast computation of matrix-type Padé approximants. *SIAM J. Matrix Analysis and Applications* 15, 3 (1994), 804–823. https://doi.org/10.1137/S0895479892230031

[3] A. Bostan, C.-P. Jeannerod, C. Mouilleron, and É. Schost. 2017. On matrices with displacement structure: generalized operators and faster algorithms. *SIAM J. on Matrix Analysis and Applications* 38, 3 (2017), 733–775. https://doi.org/10.1137/16M1062855

[4] W. Eberly, M. Giesbrecht, P. Giorgi, A. Storjohann, and G. Villard. 2007. Faster inversion and other black box matrix computation using efficient block projections. In *Proc. ISSAC*. ACM Press, 143–150. https://doi.org/10.1145/1277548.1277569

[5] P. Giorgi, C. Jeannerod, and G. Villard. 2003. On the complexity of polynomial matrix computations. In *Proc. ISSAC* (Philadelphia, PA, USA). ACM Press, 135–142. https://doi.org/10.1145/860854.860889

[6] Somit Gupta, Soumojit Sarkar, Arne Storjohann, and Johnny Valeriote. 2012. Triangular x-basis decompositions and derandomization of linear algebra algorithms over K[x]. *Journal of Symbolic Computation* 47, 4 (2012), 422–453. https://doi.org/10.1016/j.jsc.2011.09.006

[7] G. Heinig, P. Jankowski, and K. Rost. 1988. Fast inversion algorithms of Toeplitz-plus-Hankel matrices. *Numer. Math.* 52, 6 (1988), 665–682.

[8] Georg Heinig and Karla Rost. 1984. *Algebraic Methods for Toeplitz-like Matrices and Operator.* Springer, Birkhäuser Basel. https://doi.org/10.1007%2F978-3-0348-6241-7

[9] G. Heinig and K. Rost. 2004. New fast algorithms for Toeplitz-plus-Hankel matrices. *SIAM J. Matrix Analysis and Applications* 25, 3 (2004), 842–857. https://doi.org/10.1137/S0895479802410074

[10] C.-P. Jeannerod, V. Neiger, and G. Villard. 2020. Fast computation of approximant bases in canonical form. *J. Symb. Comput.* 98 (2020), 192–224. https://doi.org/10.1016/j.jsc.2019.07.011

[11] T. Kailath. 1980. *Linear Systems.* Prentice-Hall.

[12] T. Kailath, S.Y. Kung, and M. Morf. 1979. Displacement ranks of matrices and linear equations. *J. Mathematical Analysis and Applications* 68, 2 (1979), 395–407. https://doi.org/10.1016/0022-247X(79)90124-0

[13] E. Kaltofen. 1994. Asymptotically fast solution of Toeplitz-like singular linear systems. In *Proc. ISSAC* (Oxford, UK). ACM Press, 297–304. https://doi.org/10.1145/190347.190431

[14] E. Kaltofen. 2000. Challenges of symbolic computation: my favorite open problems. *J. Symbolic Computation* 29, 6 (2000), 891–919. https://doi.org/10.1006/jsco.2000.0370

[15] E. Kaltofen and B.D. Saunders. 1991. On Wiedemann's method of solving sparse linear systems. In *Proc. AAECC-9 (LNCS 539, Springer Verlag)*. 29–38.

[16] E. Kaltofen and G. Villard. 2005. On the complexity of computing determinants. *Comput. Complex.* 13, 3 (2005), 91–130. https://doi.org/10.1007/s00037-004-0185-3

[17] E. Kaltofen and G. Yuhasz. 2013. On the matrix Berlekamp-Massey algorithm. *ACM Trans. Algorithms* 9, 4 (2013), 33:1–33:24. https://doi.org/10.1145/2500122

[18] F. Le Gall. 2014. Powers of Tensors and Fast Matrix Multiplication. In *Proc. ISSAC* (Kobe, Japan). ACM Press, 296–303. https://doi.org/10.1145/2608628.2608664

[19] Victor Y. Pan. 2001. *Structured Matrices and Polynomials: Unified Superfast Algorithms.* Springer-Verlag, Berlin, Heidelberg.

[20] A. Storjohann. 2003. High-order lifting and integrality certification. *J. Symb. Comput.* 36, 3-4 (2003), 613–648. https://doi.org/10.1016/S0747-7171(03)00097-X

[21] M. Van Barel and A. Bultheel. 1992. A general module theoretic framework for vector M-Padé and matrix rational interpolation. 3 (1992), 451–462. https://doi.org/10.1007/BF02141952

[22] G. Villard. 1997. *A study of Coppersmith's block Wiedemann algorithm using matrix polynomials.* RR 975 IM IMAG. http://perso.ens-lyon.fr/gilles.villard/BIBLIOGRAPHIE/PDF/rr0497.pdf

[23] G. Villard. 2018. On computing the resultant of generic bivariate polynomials. In *Proc. ISSAC* (New York, NY, USA). ACM Press, 391–398. https://doi.org/10.1145/3208976.3209020